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A CONTINUUM DAMAGE MODEL FOR VISCOELASTIC  
MATERIALS

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A CONTINUUM DAMAGE MODEL  
FOR VISCOELASTIC MATERIALS

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Abstract

This paper presents a continuum damage model for viscoelastic materials. "Damage" is expressed by two symmetric, second-rank tensors which are related to the total areas of "active" and "passive" micro-cracks within a representative volume element of the multi-fractured material. Viscoelasticity is introduced through scalar-valued internal state variables that represent the internal degrees of freedom associated with the motions of long-chain polymeric molecules. The constitutive relations are established from basic considerations of continuum mechanics and irreversible thermodynamics, with detailed expressions derived for the case of initially isotropic materials. It is shown that damage causes softening of the material moduli as well as changes in material symmetry. The special cases of uni-axial damage under uni-axial stress and the interaction of damage with moisture diffusion are also considered.

## 1. Introduction

Continuum damage modelling, pioneered by Kachanov (1958), has become a most active subject of research in the recent decade. A comprehensive review of work in this area was given recently by Krajcinovic (1986).

Thus far, most of the modelling concerned the behavior materials like rock and concrete which exhibit brittle behavior, or metals - which creep upon reaching a plastic range. Very little attention was paid to the modelling of damage in polymeric materials that creep viscoelastically. In a recent work by Schapery (1980), viscoelastic stress-strain relations were modified to include the effects of damage. However, since those relations were not derived from a free energy and the tensorial character of the damage variable was not identified, the above approach could not specify the changes in material symmetry which may occur in damaged configurations.

Viscoelastic response in the presence of distributed micro-flaws occurs in solid propellants and is likely to be encountered in thermoplastic composites. In contrast to rock and metals, polymeric materials absorb various kinds of solvents, which may damage the polymeric composites in a variety of ways. In the case of water absorption by epoxy-based composites, such damage was noted by several investigators (e.g. Ashbee and Wyatt (1969), Shirrell et.al. (1979), Drzal et.al. (1985), and Jackson and Weitsman (1985)). Coupling between damage and diffusion was modelled by Weitsman (1987d) assuming elastic response.

In section 2 of the present work the choice of internal state variables to model damage is motivated by micromechanical considerations. Consequently "damage" is represented by two symmetric second rank tensors. This representation resembles the form employed by Murakami and Ohno (1981) - although their choice derived from different physical and mathematical

considerations.

In section 3, viscoelasticity is introduced by means of scalar valued internal state variables which correspond to the degrees of freedom afforded by the motions of molecular chains within the bulk polymer. The thermodynamic theory is then applied when this set of variables is present in addition to the tensor-valued damage state variables.

In section 4, the special case of an initially isotropic medium is considered. Viscoelastic constitutive relations are formulated, accounting for both damage growth and contraction.

Some specific circumstances are considered in section 5. These include the case of parallel, uni-directional damage and stress and the interaction between moisture flux and damage in the absence of stresses.

## 2. Basic Equations, Damage and Internal State Variables

When a solid body absorbs solvent from the exterior the mass contained within the material volume of the solid is not conserved and the solid-solvent mixture is a thermodynamically open system. The basic equations for this circumstance were developed by several researchers, including Prigogine (1968), Biot (1973), Rice and Cleary (1976) and Kestin (1979). A version of these equations was given recently by Weitsman (1987a,b). Accordingly, the "reduced entropy inequality" is expressed as follows\*:

$$\rho_s \dot{\phi} - \epsilon_{ij} \dot{\sigma}_{ij} - \rho_s \dot{s} - (q_i/T) g_i + \dot{\mu} m - \mu_{,i} f_i - g_i f_i \tilde{s} \geq 0 \quad (1)$$

In eqn. (1)  $\phi$  is the Gibbs free energy,  $\rho_s$  - the mass density of the solid,  $\epsilon_{ij}$  - components of the infinitesimal strain tensor,  $\sigma_{ij}$  - components

\*Eqn. (1) follows an earlier expression [Weitsman, 1987a, eqn (17)] upon considering infinitesimal deformations and  $\rho_s \dot{\phi} = \rho_s \dot{\psi} - \sigma_{ij} \epsilon_{ij}$ .

of Cauchy stress,  $s$  - entropy density of the solid-solvent mixture,  $T$  - temperature,  $q_i$  - components of the heat flux vector,  $g_i = \partial T / \partial x_i$ ,  $m$  - solvent mass,  $f_i$  - components of the flux of  $m$ ,  $\mu$  and  $\tilde{s}$  - the chemical potential and entropy of the solvent in a reservoir in thermodynamic equilibrium with the solid-solvent mixture.

To motivate the choice of damage variables, consider a statistically representative volume element (RVE) containing  $K$  microcracks, which may be partially or completely closed. If the material consists of several phases, then consider their equivalent homogeneous properties.

Let  $S_k$  denote the open ("active") surface of the  $k$ th micro-crack, and  $S_k^*$  its closed ("passive") surface, as shown in Fig. 1. The "macro"-level stresses and strains are quantities averaged over the volume of the RVE. Assuming infinitesimal deformations, these "macro" quantities are given by the following, well known expressions:

$$\begin{aligned} \bar{\sigma}_{ij} = \frac{1}{V} \int_V \sigma_{ij}(\underline{x}) dV = \frac{1}{2V} \left\{ \int_{S_0} (T_i^0 x_j + T_j^0 x_i) dS \right. \\ \left. + \sum_{k=1}^K \int_{S_k \cup S_k^*} (T_i^{(k)} x_j + T_j^{(k)} x_i) dS \right\} \end{aligned} \quad (2)$$

and

$$\begin{aligned} \bar{\epsilon}_{ij} = \frac{1}{V} \int_V \epsilon_{ij}(\underline{x}) dV = \frac{1}{2V} \left\{ \int_{S_0} (u_i n_j^0 + u_j n_i^0) dS \right. \\ \left. + \sum_{k=1}^K \int_{S_k \cup S_k^*} (u_i n_j^{(k)} + u_j n_i^{(k)}) dS \right\} \end{aligned} \quad (3)$$

In eqns. (2) and (3)  $S_0$  denotes the outer surface of the RVE,  $T_i$  denote tractions,  $n_i$  are components of the outward unit normal, and  $u_i$  are displacements.



If the tractions  $T_i^{(k)}$  are continuous across all surfaces  $S_k$  and  $S_k^*$ , and  $T_i^0$  correspond to a uniform stress field  $\sigma_{ij}^0$ , namely  $T_i^0 = \sigma_{ij}^0 n_j(S_0)$ , then it is well known that  $\bar{\sigma}_{ij} = \sigma_{ij}^0$ .

Let  $\Delta u_n$  and  $\Delta u_s$  denote normal and tangential components of displacement discontinuity at points on some surface  $S$ . Then, by hypothesis, on all surfaces  $S_k^*$   $\Delta u_n^{(k)} = 0$ , while  $\Delta u_s^{(k)}$  occurs against the resistance of frictional forces. It is therefore reasonable to expect that  $\Delta u_s$  on the surfaces  $S_k^*$  are much smaller than  $\Delta u_s$  on surfaces  $S_k$ .

Consequently, it is assumed that, instead of eqn. (3),  $\bar{\epsilon}_{ij}$  is given by

$$\bar{\epsilon}_{ij} = \frac{1}{2V} \left\{ \int_{S_0} (u_i n_j^0 + u_j n_i^0) dS + \sum_{k=1}^K \int_{S_k} (u_i n_j^{(k)} + u_j n_i^{(k)}) dS \right\} \quad (3a)$$

Although  $S_k^*$  no longer appear explicitly in eqn. (3a), these surfaces of discontinuity still affect  $\bar{\epsilon}_{ij}$  through an implicit presence in  $u_i$ .

To amplify this point, consider linear elastic response. In this case  $\bar{\sigma}_{ij}$  and  $\bar{\epsilon}_{ij}$  in the RVE can be constructed by superposition of:

- (a)  $T_i^0 = \sigma_{ij}^0 n_j(S_0)$  acting on the intact (undamaged) RVE, and
- (b)  $T_i^0 = 0$  on  $S_0$ ,  $T_i^{(k)} = -\sigma_{ij}^0 n_j^{(k)}$  on  $S_k$ , while on  $S_k^*$   $\Delta T_i^{(k)} = 0$ ,  $\Delta u_n^{(k)} = 0$ , and  $\Delta u_s^{(k)}$  are related linearly to the tractions. ( $k=1,2,\dots,K$ ).

Since in part (b) all tractions across  $S_k \cup S_k^*$  are continuous, obviously  $\bar{\sigma}_{ij} = \sigma_{ij}^0$  for the superposed solution.

In view of the linearity of the problem, all displacements are proportional to  $\sigma_{ij}^0$ . Hence for part (a)

$$u_i^{(a)}(\underline{x}) = S_{ijk\ell}^0 x_j \sigma_{k\ell}^0 \quad (4a)$$

while for part (b)

$$u_i^{(b)}(\underline{x}) = P_{ikl}(\underline{x}, S_1, S_2, \dots, S_K, S_1^*, S_2^*, \dots, S_K^*) \sigma_{kl}^o \quad (4b)$$

In (4a)  $S_{ijkl}^o$  denote the compliances of the undamaged material.

Substitution of eqns. (4) into eqn. (3a) yields

$$\begin{aligned} \bar{\epsilon}_{ij} = & \{ S_{ijkl}^o + \frac{1}{2V} \sum_{k=1}^K \int_{S_k} (P_{ikl}(\underline{x}, S_1, \dots, S_K, S_1^*, \dots, S_K^*) n_j^{(k)} \\ & + P_{jkl}(\underline{x}, S_1, \dots, S_K, S_1^*, \dots, S_K^*) n_i^{(k)}) dS \} \bar{\sigma}_{kl} \end{aligned} \quad (5)$$

Eqn. (5) can be expressed formally as

$$\bar{\epsilon}_{ij} = S_{ijkl}(S_1, \dots, S_K; S_1^*, \dots, S_K^*) \bar{\sigma}_{kl} \quad (6)$$

where  $S_{ijkl}$  are the damage affected compliances. However, eqn. (5) provides an additional insight by implying that changes in material symmetry are mostly due to the configuration of "active" cracks, while both "active" and "passive" cracks contribute to the overall softening. The above assumption simplifies the constitutive expressions developed in section 4. However, it should be emphasized that the general development of the constitutive formalism does not depend on the assumptions that led to the approximation expressed in eqn. (5).

It seems that a complete accounting of the response of the RVE requires the incorporation of  $2K$  parameters which represent the surfaces  $S_k$  and  $S_k^*$  ( $k=1, \dots, K$ ). Since the shapes of those surfaces are generally unknown, they may be approximated by some equivalent flat surfaces, namely by vectors  $\underline{D}^{(k)} = D^{(k)} \underline{n}^{(k)}$  and  $\underline{D}^{*(k)} = D^{*(k)} \underline{n}^{(k)}$  (no sum on  $k$ ,  $k=1, \dots, K$ ). However, since each microcrack has two equal and opposite surfaces, the representation should be independent of the sense of  $\underline{n}^{(k)}$ . Following Spencer (1962), this is accomplished by representing each microcrack by the symmetric dyad  $\underline{A}^{(k)} = \underline{D}^{(k)} \underline{D}^{(k)}$ . Similarly we have  $\underline{A}^{*(k)} = \underline{D}^{*(k)} \underline{D}^{*(k)}$ .

Due to the paucity of detailed information regarding the size and



location of the surfaces  $S_k$  and  $S_k^*$  it is proposed to account for their joint effect on  $P_{ikl}$  in eqn. (4b) and  $S_{ijkl}$  in eqn. (6) by means of their dyadic sums, namely through

$$\underline{\underline{A}} = \sum_{k=1}^K \underline{\underline{D}}^{(k)} \underline{\underline{D}}^{(k)}$$

and

$$\underline{\underline{A}}^* = \sum_{k=1}^K \underline{\underline{D}}^{*(k)} \underline{\underline{D}}^{*(k)}$$

(7)

Hence, the effects of "damage" are represented by two, "macro"-level, internal state variables that are symmetric, second-rank tensors.

The selection of  $\underline{\underline{A}}$  to represent damage was recently employed by Weitsman (1987c). It was also alluded to in an earlier work by Kachanov (1980).

The tensors  $\underline{\underline{A}}$  and  $\underline{\underline{A}}^*$  can be non-dimensionalized through division by a characteristic area  $\Delta$ , e.g. any one of the "walls" of the RVE. This leads to the non-dimensional quantities

$$\underline{\underline{a}} = \underline{\underline{A}}/\Delta^2, \quad \underline{\underline{a}}^* = \underline{\underline{A}}^*/\Delta^2 \quad (8)$$

Although the present work considers infinitesimal strains ( $\epsilon_{ij} \ll 1$ ), the abovementioned representation of "damage" does not infer small  $a_{ij}$  and  $a_{ij}^*$ . Consequently, Taylor series expansions are useful in  $\epsilon_{ij}$  but not in  $a_{ij}$  and  $a_{ij}^*$ .

As mentioned earlier (Weitsman, 1987c) the present choice of damage variables yields, in the case of randomly oriented micro-cracks of equal area, the intuitively appealing result  $a_{ij} = a^2 \delta_{ij}$  in both two and three dimensions. This outcome, which implies that randomly distributed damage affects the material's response as a scalar quantity devoid of orientation, is not borne out when a vectorial representation  $\underline{\underline{D}} = D_n$  is chosen for "damage".

Viscoelastic behavior, which occurs most commonly in polymeric materials, has been attributed to the various degrees of freedom afforded by the motions of intertwined long-chain molecules. Such considerations led Rouse (1953), Bueche (1954), and Zimm (1956), to express the motions of molecular chains consisting of  $N$  segments in forms which, upon transformation to principal coordinates and normal modes, resulted in  $N$  scalar-valued internal state variables. These formulations motivated subsequent researchers (Biot 1954a,b, Schapery 1964, 1966, 1969) to derive a theory of viscoelasticity from fundamental concepts of irreversible thermodynamic processes, with irreversibility stemming from  $N$  internal state variables whose "growth laws" derived from the motion of chain segments subjected to viscous resistance.

It is worth noting that in the abovementioned works the tensorial character of the internal state variables was not specified and stress-strain relations for viscoelastic materials of various symmetries were inferred from analogies with the elastic case. This procedure will not carry over to the present work, which incorporates "damage" by means of two symmetric second-rank tensors. As shown in sections 3 and 4, the derivation of constitutive expressions and symmetry relations hinges on the identification of the viscoelastic internal state variables as scalars.

### 3. Constitutive Relations: Thermodynamic Considerations.

Consider a viscoelastic material with continuously distributed damage represented by the symmetric second rank tensors  $a_{ij}$ ,  $a_{ij}^*$  defined in section 2. Furthermore let  $\gamma_r$  ( $r=1, \dots, N$ ) denote  $N$  scalar-valued internal state variables which represent the internal degrees of freedom of molecular motion within the viscoelastic polymer.

The subsequent formulation is based upon the premise that the internal

molecular motions, represented by  $\gamma_r$ , occur on a dimensional scale which is much smaller than the micro-damage expressed by  $a_{ij}$  and  $a_{ij}^*$ . This suggests that the viscoelastic retardation times are unaffected by damage, although damage growth rates are influenced by the inherent viscoelasticity of the material.

To maintain tractability, and focus on the effects of damage, we shall assume a common time dependence for all compliances. This assumption is valid for many isotropic polymers which exhibit a constant Poisson's ratio.

Accordingly, consider a Gibbs free energy  $\phi$  of the form

$$\phi = f(\sigma_{ij}, a_{ij}, a_{ij}^*, m, T) \phi(m, T, \gamma_r) \quad (r=1, \dots, N) \quad (9)$$

Without loss of generality let  $f > 0$ . Also, let  $\phi$  be dimensionless.

Consider a representative volume element subjected to fixed levels of  $\sigma_{ij}$ ,  $m$ , and  $T$ . Following Coleman and Gurtin (1967), this circumstance will trigger an irreversible thermodynamic process in the material, causing the internal variables  $a_{ij}$ ,  $a_{ij}^*$  and  $\gamma_r$  to drift towards their equilibrium values  $a_{ij}^e$ ,  $a_{ij}^{*e}$  and  $\gamma_r^e$ .

According to the extremum principles of thermodynamics (Callen, 1960) the Gibbs free energy attains a minimum at  $a_{ij}^e$ ,  $a_{ij}^{*e}$  and  $\gamma_r^e$ . Therefore, at those values we have

$$\frac{\partial \phi}{\partial \gamma_r} = 0, \quad \frac{\partial \phi}{\partial a_{ij}} = 0, \quad \frac{\partial \phi}{\partial a_{ij}^*} = 0 \quad (10)$$

and

$$\begin{aligned} & \frac{1}{2} \frac{\partial^2 \phi}{\partial \gamma_r \partial \gamma_s} \delta \gamma_r \delta \gamma_s + \frac{\partial^2 \phi}{\partial \gamma_r \partial a_{ij}} \delta \gamma_r \delta a_{ij} + \frac{\partial^2 \phi}{\partial \gamma_r \partial a_{ij}^*} \delta \gamma_r \delta a_{ij}^* \\ & + \frac{1}{2} \frac{\partial^2 \phi}{\partial a_{ij} \partial a_{kl}} \delta a_{ij} \delta a_{kl} + \frac{\partial^2 \phi}{\partial a_{ij} \partial a_{kl}^*} \delta a_{ij} \delta a_{kl}^* + \frac{1}{2} \frac{\partial^2 \phi}{\partial a_{ij}^* \partial a_{kl}^*} \delta a_{ij}^* \delta a_{kl}^* > 0 \end{aligned} \quad (11)$$

Eqn. (11) holds at  $a_{ij}^e, a_{ij}^{*e}, \gamma_r^e$  for all arbitrary variations  $\delta a_{ij}, \delta a_{ij}^*$  and  $\delta \gamma_r$ .

Expanding  $\phi$  about  $a_{ij}^e, a_{ij}^{*e}, \gamma_r^e$  employing eqns. (9) and (10) and assuming  $\gamma_r < \gamma_r^e \ll 1$  we get

$$\phi(\sigma_{ij}; m, T, a_{ij}^e, a_{ij}^{*e}, \gamma_r) = \phi^e + \frac{1}{2} f(\sigma_{ij}, a_{ij}^e, a_{ij}^{*e}, m, T) \frac{\partial^2 \phi}{\partial \gamma_r \partial \gamma_s} \cdot (\gamma_r - \gamma_r^e)(\gamma_s - \gamma_s^e) + \text{higher order terms in } (\gamma_r - \gamma_r^e) \quad (12)$$

Alternately, an expansion about  $a_{ij} = a_{ij}^e, a_{ij}^* = a_{ij}^{*e}$  and  $\gamma_r = 0$  gives

$$\begin{aligned} \phi(\sigma_{ij}, m, T, a_{ij}^e, a_{ij}^{*e}, \gamma_r) &= \phi(\sigma_{ij}, a_{ij}^e, a_{ij}^{*e}, m, T, 0) \\ &+ f(\sigma_{ij}, a_{ij}^e, a_{ij}^{*e}, m, T) \frac{\partial \phi}{\partial \gamma_r} \Big|_{\gamma_r=0} + \frac{1}{2} f(\sigma_{ij}, a_{ij}^e, a_{ij}^{*e}, m, T) \frac{\partial^2 \phi}{\partial \gamma_r \partial \gamma_s} \Big|_{\gamma_r=0} \gamma_r \gamma_s \\ &+ \text{higher order terms in } \gamma_r \end{aligned} \quad (13)$$

In view of the assumption that  $(\gamma_r, \gamma_r^e) \ll 1$  expansions (12) and (13) are equal, therefore

$$\begin{aligned} \phi(\sigma_{ij}, m, T, a_{ij}^e, a_{ij}^{*e}, 0) &= \phi^e \\ &+ \frac{1}{2} f(\sigma_{ij}, a_{ij}^e, a_{ij}^{*e}, m, T) \frac{\partial^2 \phi}{\partial \gamma_r \partial \gamma_s} \Big|_{\gamma_p = \gamma_p^e} \gamma_r^e \gamma_s^e \end{aligned} \quad (14a)$$

$$\frac{\partial \phi}{\partial \gamma_r} \Big|_{\gamma_p=0} = - \frac{\partial^2 \phi}{\partial \gamma_r \partial \gamma_s} \Big|_{\gamma_p = \gamma_p^e} \gamma_s^e \quad (14b)$$

and, in particular

$$\frac{\partial^2 \phi}{\partial \gamma_r \partial \gamma_s} \Big|_{\gamma_p=0} = \frac{\partial^2 \phi}{\partial \gamma_r \partial \gamma_s} \Big|_{\gamma_p = \gamma_p^e} \quad (14c)$$

Denote  $\frac{\partial \phi}{\partial \gamma_r} \Big|_{\gamma_r=0} = \phi_r$  and  $\frac{\partial^2 \phi}{\partial \gamma_r \partial \gamma_s} \Big|_{\gamma_p=0} = \phi_{rs}$ . It follows from eqns. (9), (11) and

(14a) that  $\phi_{rs} > 0$  independently of  $a_{ij}$  and  $a_{ij}^*$ .

Following the conceptual models of Rouse (1953) and Bueche (1955), and their incorporation into a growth relationship for the internal state variable as proposed by deGroot and Mazur (1962), Biot (1954a,b) and Schapery (1964), assume that  $\dot{\gamma}_s$  depend only on that portion of  $\phi$  which concerns the internal processes within the polymer (excluding damage). Thus let

$$\frac{\partial \phi}{\partial \gamma_r} = -b_{rs} \dot{\gamma}_s \quad (r, s = 1, \dots, N) \quad (15)$$

For linear viscoelastic behavior  $b_{rs} = b_{rs}(m, T)$ . Furthermore, by Onsager's principle  $b_{rs} = b_{sr}$ .

Expanding  $\phi(m, T, \gamma_r)$  in powers of  $\gamma_r$ , truncating after the second power in view of the foregoing assumption that  $\gamma_r \ll 1$ , we have

$$\phi(m, T, \gamma_r) = \phi(m, T, 0) + \phi_r(m, T, 0) \gamma_r + \frac{1}{2} \phi_{rs}(m, T, 0) \gamma_r \gamma_s \quad (16)$$

Substitution of (16) in (15) gives

$$\phi_r + \phi_{rs} \gamma_s = -b_{rs} \dot{\gamma}_s \quad (17)$$

Substitution of the specific form of  $\phi$ , given in eqn. (9), into eqn. (1) yields

$$\begin{aligned} & -\rho_s \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - \rho_s \frac{\partial f}{\partial a_{ij}} \dot{a}_{ij} - \rho_s \frac{\partial f}{\partial a_{ij}^*} \dot{a}_{ij}^* - \rho_s \frac{\partial \phi}{\partial m} \dot{m} - \rho_s \frac{\partial \phi}{\partial T} \dot{T} \\ & - \rho_s f \phi_r \dot{\gamma}_r - \epsilon_{ij} \dot{\sigma}_{ij} - \rho_s s \dot{T} - (q_i/T) g_i + \mu \dot{m} - \mu_{,i} f_{,i} - g_i f_{,i} \tilde{s} \geq 0 \end{aligned} \quad (18)$$

Since eqn. (18) cannot be violated for any process, familiar arguments (e.g. Jaunzemis 1967) give

$$\epsilon_{ij} = -\rho_s \phi \frac{\partial f}{\partial \sigma_{ij}}, \quad \mu = \rho_s \frac{\partial \phi}{\partial m}, \quad s = -\frac{\partial \phi}{\partial T} \quad (19)$$

and

$$-\rho_s \phi \frac{\partial f}{\partial a_{ij}} \dot{a}_{ij} - \rho_s \phi \frac{\partial f}{\partial a_{ij}^*} \dot{a}_{ij}^* - \rho_s f \frac{\partial \phi}{\partial \gamma_r} \dot{\gamma}_r - (q_i/T) g_i - f_{,i} \mu_{,i} - g_i f_{,i} \tilde{s} \geq 0 \quad (20)$$

In particular, at fixed damage, moisture and temperature

$$-\frac{\partial \phi}{\partial \gamma_r} \dot{\gamma}_r \geq 0 \quad (20a)$$

Equations (15) and (20a) give  $b_{rs} \dot{\gamma}_r \dot{\gamma}_s \geq 0$ , namely  $b_{rs}(m, T)$  are components of a symmetric, semi-positive definite matrix. Since a similar conclusion was reached earlier for  $\phi_{rs}$ , it is possible to diagonalize  $b_{rs}$  and  $\phi_{rs}$  simultaneously and rewrite eqn. (17) in the form

$$B_r \dot{\gamma}_r + \chi_r \gamma_r = -\phi_r \quad (r = 1, 2, \dots, N. \text{ No sum on } r) \quad (21)$$

The solution of the  $N$ , now uncoupled, equations (21) is

$$\gamma_r = K_r (1 - e^{-t/Z_r}) \quad (r = 1, \dots, N. \text{ No sum on } r) \quad (22)$$

where  $K_r = -\phi_r/\chi_r$  and  $Z_r = B_r/\chi_r > 0$ . Both  $K_r$  and  $Z_r$  depend on  $m$  and  $T$ .

As noted by Schapery (1969), if the dependence of  $b_{rs}$  on  $m$  and  $T$  is common, namely

$$b_{rs}(m, T) = b(m, T) b_{rs}^0 \quad (b_{rs}^0 \text{ constants}) \quad (23)$$

then, eqn. (17) reduces to

$$\phi_r + \phi_{rs} \gamma_s + b_{rs}^0 \frac{d}{d\psi} \gamma_s = 0 \quad (24)$$

and, instead of (22), we have the simpler result

$$\gamma_r = K_r (1 - e^{-\psi/Z_r}) \quad (25)$$

In equations (24) and (25)  $\psi = t/b(m, T)$  is the "reduced time", and  $b(m, T)$  is the moisture-and-temperature-dependent shift factor.

If, in addition, all  $\phi_{rs} = b_G(m, T) \phi_{rs}^0$ ,  $\phi_{rs}^0 = \text{constants}$ , then eqn. (17) reads



$$\phi_r + b_G \phi_{rs}^{\circ} \gamma_s + b_{rs}^{\circ} \frac{d\gamma_s}{d\psi} = 0 \quad (24a)$$

and its solution reduces to

$$\gamma_r = - \frac{\phi_r}{x_r^{\circ} b_G} (1 - e^{-\xi/Z_r^{\circ}}) \quad (26)$$

$$\text{where } \xi = b_G(m, T) \psi = \frac{b_G(m, T)}{b(m, T)} t \quad (27)$$

An expansion of  $\phi(m, T, \gamma_r)$  in powers of  $\gamma_r$  - truncating after the second power - and employing the expression given in eqn. (26), yields the form\*

$$\phi = \bar{\phi} \frac{L_r}{b_G} (1 - e^{-\xi/\tau_r}) \quad (28)$$

In the case of a continuous spectrum of retardation-times we obtain

$$\phi(m, T, t) = \frac{1}{b_G} \int_0^{\infty} L(\tau) (1 - e^{-\xi/\tau}) d\tau = \frac{1}{b_G} D(\xi) \quad (29)$$

To proceed beyond this stage it is necessary to consider material symmetry. Isotropy will be selected, since it applies to particle reinforced polymeric composites as well as provides the simplest illustration.

#### 4. Isotropic Constitutive Relations

In the case of isotropy  $f(\sigma_{ij}, a_{ij}, a_{ij}^*)$  depends on the isotropic invariants of these three symmetric, second-rank tensors. Following Spencer (1971) these invariants are:

$$\begin{aligned} I_1 &= \sigma_{kk}, I_2 = \sigma_{ij} \sigma_{ji}, I_3 = \sigma_{ij} \sigma_{jk} \sigma_{ki}, I_4 = a_{kk}, I_5 = a_{ij} a_{ji}, I_6 = a_{ij} a_{jk} a_{ki}, \\ I_7 &= I_4^*, I_8 = I_5^*, I_9 = I_6^*, I_{10} = a_{ij} \sigma_{ij}, I_{11} = a_{ij} a_{jk} \sigma_{ki}, I_{12} = a_{ij} \sigma_{jk} \sigma_{ki}, \\ I_{13} &= a_{ij} a_{jk} \sigma_{kl} \sigma_{li}, I_{14} = a_{ij} a_{ji}^*, I_{15} = a_{ij} a_{jk} a_{ki}^*, I_{16} = a_{ij} a_{jk}^* a_{ki}^* \end{aligned}$$

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\*Note that this form is retained even for products of  $\gamma_r$  because products like  $\prod_{k=1}^n (1 - e^{-a_k x})$  can be expressed as sums  $\sum_{k=1}^n (1 - e^{-a_k x})$ .

$I_{17} = a_{ij}a_{jk}a_{kl}^*a_{li}^*$  and nine more invariants which include  $\sigma_{ij}$  and  $a_{kl}^*$  (with or without  $a_{mn}$ ). In the above list  $I^* = I(a_{ij}^*)$ . The nine unlisted invariants are discarded in  $f$  because it is assumed herein that, while both "active" and "passive" microcracks participate in the softening of the material, only the active cracks affect its symmetry. This assumption is consistent with the earlier premises indicated in equations (3a) and (5).

Expanding  $f(\sigma_{ij}, a_{ij}, a_{ij}^*)$  in powers of  $\sigma_{ij}$ , terminating at the second power, we obtain:

$$\begin{aligned} -\rho_s f(\sigma_{ij}, a_{ij}, a_{ij}^*, m, T) = & -\rho_s A_0 + A_1 \sigma_{kk} + A_2 a_{ij} \sigma_{ij} + A_3 a_{ij} a_{jk} \sigma_{ki} + \frac{1}{2} A_4 \sigma_{kk} \sigma_{ll} \\ & + \frac{1}{2} A_5 \sigma_{ij} \sigma_{ji} + A_6 a_{ij} \sigma_{ij} \sigma_{kk} + A_7 a_{ij} a_{jk} \sigma_{ki} \sigma_{ll} \\ & + A_8 a_{ij} \sigma_{ij} a_{kl} a_{lm} \sigma_{mk} + \frac{1}{2} A_9 a_{ij} \sigma_{ij} a_{kl} \sigma_{kl} \\ & + \frac{1}{2} A_{10} a_{ij} a_{jk} \sigma_{ki} a_{lm} a_{mn} \sigma_{nl} + A_{11} a_{ij} \sigma_{jk} \sigma_{ki} \\ & + A_{12} a_{ij} a_{jk} \sigma_{kl} \sigma_{li} \end{aligned} \quad (30)$$

In eqn. (30) the terms  $A_0, A_1, \dots, A_{12}$  are functions of the damage invariants  $I_4 - I_9$  and  $I_{14} - I_{17}$ ,  $m$ , and  $T$ .

Strain-stress relations are obtained from eqn. (19)<sub>1</sub>. In view of eqn. (30) we have

$$\begin{aligned} \epsilon_{pq}^{el} = - \frac{\partial \rho_s f}{\partial \sigma_{pq}} = & A_1 \delta_{pq} + A_2 a_{pq} + A_3 a_{qj} a_{jp} + A_4 \sigma_{kk} \delta_{pq} + A_5 \sigma_{pq} \\ & + A_6 (a_{ij} \sigma_{ij} \delta_{pq} + a_{pq} \sigma_{kk}) + A_7 (a_{qj} a_{jp} \sigma_{kk} + a_{ij} a_{jk} \sigma_{ki} \delta_{pq}) \\ & + A_8 (a_{pq} a_{ij} a_{jk} \sigma_{ki} + a_{ij} \sigma_{ij} a_{qk} a_{kp}) \\ & + A_9 a_{pq} a_{ij} \sigma_{ij} + A_{10} a_{qj} a_{jp} a_{ik} a_{kl} \sigma_{li} \\ & + A_{11} (a_{ip} \sigma_{qi} + a_{qi} \sigma_{ip}) + A_{12} (a_{ij} a_{jp} \sigma_{qi} + a_{qj} a_{ji} \sigma_{ip}) \end{aligned} \quad (31)$$

The first three terms on the right side on eqn. (31) are related to hygrothermal expansion in the absence of stress. In addition,  $A_1$ ,  $A_4$  and  $A_5$  should reduce to  $(\alpha\Delta T + \beta\Delta m)$  and to the "classical" compliances  $J_1$  and  $J_2$  in the absence of damage.

It can be noted from eqn. (31) that  $\epsilon_{pq}^{el} = \epsilon_{qp}^{el}$ . Furthermore, if all active microcracks are parallel to a common plane it can be shown that eqn. (31) corresponds to transverse isotropy in that plane. Similarly, if all active microcracks are parallel to two or three mutually perpendicular planes eqn. (31) corresponds, as expected, to orthotropic symmetry.

For fixed  $a_{ij}$ ,  $a_{ij}^*$ ,  $\sigma_{ij,m}$ , and  $T$  the viscoelastic strain-strain relations are simply a product of expressions (29) and (31), namely

$$\epsilon_{pq}(t) = \epsilon_{pq}^{el} \cdot D(\xi)/b_G \quad (32)$$

For fixed damage but for fluctuating  $\sigma_{ij}$ ,  $m$ , and  $T$  the straightforward employment of superposition integrals gives

$$\begin{aligned} b_G(m(t), T(t)) \epsilon_{pq}(t) = & (\alpha_1 \delta_{pq} + \alpha_2 a_{pq} + \alpha_3 a_{qj} a_{jp}) \int_0^t D(\xi - \xi') \frac{\partial \Delta T}{\partial \tau} d\tau \\ & + (\beta_1 \delta_{pq} + \beta_2 a_{pq} + \beta_3 a_{qj} a_{jp}) \int_0^t D(\xi - \xi') \frac{\partial \Delta m}{\partial \tau} d\tau \\ & + A_4 \delta_{pq} \int_0^t D(\xi - \xi') \frac{\partial \sigma_{kk}}{\partial \tau} d\tau + A_5 \int_0^t D(\xi - \xi') \frac{\partial \sigma_{pq}}{\partial \tau} d\tau + \dots \\ & + A_{12} \{ a_{ij} a_{jp} \int_0^t D(\xi - \xi') \frac{\partial \sigma_{qi}}{\partial \tau} d\tau + a_{qj} a_{ji} \int_0^t D(\xi - \xi') \frac{\partial \sigma_{ip}}{\partial \tau} d\tau \} \end{aligned} \quad (33)$$

In eqn. (33),  $\alpha_i$  and  $\beta_i$  ( $i=1,2,3$ ), as well as in all  $A_j$  ( $j=4, \dots, 12$ ), depend on the invariants  $I_4 - I_9$ ,  $I_{14} - I_{17}$  and, possibly, on  $m$  and  $T$ . Also

$$\xi = \xi(t) = \int_0^t \frac{du}{b[m(u), T(u)]} \quad \text{and} \quad \xi' = \xi(\tau).$$

Regarding the case of fluctuating damage, consider first the situation of

monotonic growth of all micro-cracks. In this circumstance all  $S_k^* = 0$  in eqn. (6) and the compliances are  $S_{ijkl} = S_{ijkl}(S_1(t), S_2(t), \dots S_K(t))$ . In the context of the continuum damage variables employed herein we have  $S_{ijkl}(t) = S_{ijkl}(a_{mn}(t))$ . Under constant  $\sigma_{ij}$ ,  $m$ , and  $T$ , and for the elastic case, eqn. (6) reads

$$\epsilon_{ij}(t) = S_{ijkl}(a_{pq}(t)) \sigma_{kl}^0$$

which can be expressed as

$$\epsilon_{ij}(t) = S_{ijrs}(a_{pq}(0)) C_{rsmn}(a_{pq}(0)) S_{mnkl}(a_{pq}(t)) \sigma_{kl}^0 \quad (34)$$

Eqn. (34), where  $C_{ijkl} = S_{ijkl}^{-1}$ , shows that the case of growing damage can be viewed formally as that of fixed damage with an appropriately modified, time-varying stress.

In view of the assumed monotonic growth, upon postulating continuity of  $S_{ijkl}$  in  $a_{pq}$ , the entire history-dependence of  $S_{ijkl}$  is given by

$$S_{ijkl}(a_{pq}(t)) = S_{ijkl}(a_{pq}(0)) + \int_0^t \frac{\partial S_{ijkl}(a_{pq})}{\partial a_{mn}}(\tau) \frac{\partial a_{mn}(\tau)}{\partial \tau} d\tau \quad (35)$$

The circumstances considered in eqns. (34) and (35) permit the employment of the correspondence principle (Lee, 1955).

Consequently, with  $\sigma_{ij} = \sigma_{ij}(t)$ , the strain-stress relations take the form

$$b_G(m, T) \epsilon_{ij}(t) = S_{ijkl}(a_{mn}(0)) \int_0^t D(\xi - \xi') \frac{\partial \sigma_{kl}}{\partial \tau} d\tau + \int_0^t D(\xi - \xi') \frac{\partial}{\partial \tau} [\sigma_{kl}(\tau) \int_0^\tau \frac{\partial S_{ijkl}(a_{mn})}{\partial a_{mn}} \frac{\partial a_{mn}(u)}{\partial u} du] d\tau \quad (36)$$

Upon identifying  $S_{ijkl}(a_{pq})$  with the compliances and forms for the initially isotropic case in eqn. (31), the strain-stress relations, with monotonically growing microcracks, and time-dependent  $\sigma_{ij}$ ,  $m$ , and  $T$ , read

$$\begin{aligned}
b_G \epsilon_{pq}(t) = D^* \{ & \Delta T \circ (\alpha_1 \delta_{pq} + \alpha_2 a_{pq} + \alpha_3 a_{qj} a_{jp}) \\
& + \Delta m \circ (\beta_1 \delta_{pq} + \beta_2 a_{pq} + \beta_3 a_{qj} a_{jp}) + (\sigma_{kk} \circ A_4) \delta_{pq} \\
& + (\sigma_{pq} \circ A_5) + [\sigma_{ij} \circ (A_6 a_{ij})] \delta_{pq} + (\sigma_{kk} \circ (A_6 a_{pq})) \\
& + (\sigma_{kk} \circ (A_7 a_{qj} a_{jp})) + (\sigma_{ki} \circ (A_7 a_{ij} a_{jk})) \delta_{pq} \\
& + (\sigma_{ki} \circ (A_8 a_{pq} a_{ij} a_{jk})) + (\sigma_{ij} \circ (A_8 a_{ij} a_{qk} a_{kp})) \\
& + (\sigma_{ij} \circ (A_9 a_{pq} a_{ij})) + (\sigma_{li} \circ (A_{10} a_{qj} a_{jp} a_{ik} a_{kl})) \\
& + (\sigma_{qi} \circ (A_{11} a_{ip})) + (\sigma_{ip} \circ (A_{11} a_{qi})) + (\sigma_{qi} \circ (A_{12} a_{ij} a_{jp})) \\
& + (\sigma_{ip} \circ (A_{12} a_{qj} a_{ji})) \} \quad (37)
\end{aligned}$$

where  $D^* f$  denotes  $\int_0^t D(\xi - \xi') \frac{\partial f}{\partial \tau} d\tau$  and  $(g \circ h)(\tau)$  denotes

$$g(\tau) [h(a_{mn}(0)) + \int_0^\tau \frac{\partial h}{\partial a_{mn}} \frac{\partial a_{mn}}{\partial u} du]. \quad (38)$$

Obviously, in eqn. (37) all  $\alpha_i$ ,  $\beta_i$  and  $A_j$  ( $i=1,2,3$ ;  $j=4,5,\dots,12$ ) depend only on the invariants  $I_4$ ,  $I_5$  and  $I_6$  of  $a_{ij}$ .

In particular, if the material is initially undamaged then  $\alpha_1(a_{mn}(0)) = \alpha$ ,  $\beta_1(a_{mn}(0)) = \beta$ ,  $A_4(a_{mn}(0)) = J_1$  and  $A_5(a_{mn}(0)) = J_2$  which denote the familiar viscoelastic expansional coefficients and compliances. In this case the initial values of all other terms -  $\alpha_2$ ,  $\alpha_3$ ,  $\beta_2$ ,  $\beta_3$ ,  $A_6$ , ...  $A_{12}$  - vanish.

When the micro-cracks experience both growth and closure the current configuration, as given by  $a_{ij}(t)$  and  $a_{ij}^*(t)$ , may reflect distinct growth and closure histories. The viscoelastic strain-stress relation should distinguish among such histories.

Considering an initially undamaged material, the stress-strain relations under damage growth and closure read:

$$\begin{aligned}
b_G \epsilon_{ij}(t) = & S_{ijkl}(0) \int_0^t D(\xi - \xi') \frac{\partial \sigma_{kl}}{\partial \tau} d\tau \\
& + \int_0^t D(\xi - \xi') \frac{\partial}{\partial \tau} [\sigma_{kl}(\tau) \int_0^\tau (g) \frac{\partial S_{ijkl}(a_{mn}, a_{pq}^*)}{\partial a_{mn}} \frac{\partial a_{mn}}{\partial u} du] d\tau
\end{aligned}$$

$$+ \int_0^t D(\xi - \xi') \frac{\partial}{\partial \tau} [\sigma_{kl}(\tau) \int_0^{\tau(c)} \frac{\partial S_{ijkl}(a_{mn}, a_{pq}^*)}{\partial a_{pq}^*} \frac{\partial a_{pq}^*}{\partial u} du] d\tau \quad (39)$$

where  $\tau(g)$  and  $\tau(c)$  represent the time intervals for damage growth and closure, respectively. Those intervals may over-lap.

Reverting to the initially isotropic case, the result is still given by eqn. (37), except that expression (38) should be modified to read

$$(g \circ h)(\tau) = g(\tau) [h(0) + \int_0^{\tau(g)} \frac{\partial h(a_{mn}, a_{pq}^*)}{\partial a_{mn}^*} \frac{\partial a_{mn}^*}{\partial u} du + \int_0^{\tau(c)} \frac{\partial h(a_{mn}, a_{pq}^*)}{\partial a_{pq}^*} \frac{\partial a_{pq}^*}{\partial u} du] \quad (39a)$$

## 5. Some Special Sub-Cases.

### (a) Unidirectional damage under uniaxial stress.

Let all "active" and "passive" micro-cracks be in planes normal to the  $x_3$  - direction, whereby  $a_{ij} = a \delta_{3i} \delta_{3j}$  and  $a_{ij}^* = a^* \delta_{3i} \delta_{3j}$ .

In this case eqn. (31) reduces to the transversely isotropic relation

$$\begin{aligned} \epsilon_{pq}^{el} = & J_1 \sigma_{kk} \delta_{pq} + J_2 \sigma_{pq} + A(\sigma_{33} \delta_{pq} + \delta_{3p} \delta_{3q} \sigma_{kk}) \\ & + B \delta_{3p} \delta_{3q} \sigma_{33} + C(\delta_{3p} \sigma_{3q} + \delta_{3q} \sigma_{3p}) \end{aligned} \quad (40)$$

In eqn. (40) A, B, and C represent combinations of the seven compliances  $A_6, \dots, A_{12}$  of eqn. (31). Also, all five compliances in eqn. (40) depend on a and  $a^*$ .

Consider the case of uni-axial stress  $\sigma_{33} = \sigma_0 H(t)$  and, in addition, assume  $m=0$  and  $\Delta T=0$ . In this case the strains are

$$\begin{aligned} \epsilon_{11}^{el} = \epsilon_{22}^{el} &= (J_1 + A) \sigma_0 \\ \epsilon_{33}^{el} &= (J_1 + J_2 + 2A + B + 2C) \sigma_0 \end{aligned} \quad (41)$$

In view of eqn. (39a), the corresponding viscoelastic strains are



$$\begin{aligned}
\epsilon_{11}(t)/\sigma_0 = \epsilon_{22}(t)/\sigma_0 &= J_1(0,0)D(t) + \int_0^t D(t-\tau) \left[ -\frac{\partial F_1(a, a^*)}{\partial a} \frac{\partial a(\tau(g))}{\partial \tau(g)} d\tau(g) \right. \\
&\quad \left. + \frac{\partial F_1(a, a^*)}{\partial a^*} \frac{\partial a^*(\tau(c))}{\partial \tau(c)} d\tau(c) \right] \\
\epsilon_{33}(t)/\sigma_0 &= [J_1(0,0) + J_2(0,0)]D(t) + \int_0^t D(t-\tau) \left[ -\frac{\partial F_2(a, a^*)}{\partial a} \frac{\partial a(\tau(g))}{\partial \tau(g)} d\tau(g) \right. \\
&\quad \left. + \frac{\partial F_2(a, a^*)}{\partial a^*} \frac{\partial a^*(\tau(c))}{\partial \tau(c)} d\tau(c) \right]
\end{aligned} \tag{42}$$

In eqns. (42)  $F_1 = J_1 + A$ ,  $F_2 = J_1 + J_2 + 2A + B + 2C$ , and  $\tau(g)$ ,  $\tau(c)$  denote times of crack growth and closure, respectively. These expressions show that Poisson's ratio, as inferred from  $-\epsilon_{11}(t)/\epsilon_{33}(t)$ , can serve as a measure of the effects of damage on material properties.

(b) Coupling of damage with moisture transport.

Consider the flux of moisture  $f_i$  in the presence of damage but in the absence of stress. In addition, let the temperature remain constant, i.e.  $\Delta T = 0$ .

The flux  $f_i$  will depend on the gradient of the chemical potential  $z_i = \mu/\alpha x_i$  as well as on  $a_{ij}$  and  $a_{ij}^*$ .

To derive the flux-gradient relationship it is necessary to generate all the isotropic invariants among the two vectors and two symmetric second-rank tensors  $f_i$ ,  $z_i$ ,  $a_{ij}$  and  $a_{ij}^*$ . According to Spencer (1971, page 293) there are seventeen such invariants.\* To maintain tractability we shall consider only the seven first terms from Spencer's list. Accordingly

\*Invariants of the form  $U_i(\pi_\alpha)_{ij}U_j$  are irrelevant for the construction of flux - gradient relationships.

$$\begin{aligned}
f_i = & F^{(1)} z_i + F^{(2)} a_{ij} z_j + F^{(3)} a_{ik} a_{kj} z_j + F^{(4)} a_{ij}^* z_j \\
& + F^{(5)} a_{ik}^* a_{kj}^* z_j + F^{(6)} (a_{ik} a_{kj}^* + a_{jk} a_{ki}^*) z_j \\
& + F^{(7)} (a_{ik} a_{kj}^* - a_{jk} a_{ki}^*) z_j
\end{aligned} \tag{43}$$

In eqn. (43), all  $F^{(i)}$  ( $i=1,2,\dots,7$ ) are assumed to depend on the damage invariants  $I_4 - I_9$ ,  $I_{14} - I_{17}$  listed in section 4.

According to eqn. (43) the flux of moisture is affected by changes in the material symmetry caused by both damage parameters  $a_{ij}$  and  $a_{ij}^*$ . Recall eqns. (19)<sub>2</sub>, (29) and (30). Accordingly, in the absence of stress

$$\begin{aligned}
\mu &= \frac{\partial}{\partial m} (A_0 D(\xi) / b_G) \\
&= \frac{\partial}{\partial m} (A_0 / b_G) D(\xi) + (A_0 / b_G) D'(\xi) \frac{\partial}{\partial m} \left( \frac{b_G}{b} \right) t
\end{aligned} \tag{44}$$

Since the boundary condition for the moisture transport process is  $\mu(x|_{\underline{x} \in \text{boundary}}) = \mu^A$  (Ambient vapor), eqn. (44) shows that even under constant ambient humidity, the boundary condition will be time-dependent. A similar conclusion was reached by Long and Richman (1959), Frisch (1964), Jackle and Frisch (1985), and Weitsman (1987a) - even in the absence of damage.

Consider now the gradients  $z_i = \partial \mu / \partial x_i$ . In view of eqn. (44)  $z_i$  will include gradients of damage, through the dependence of  $A_0$  on the invariants  $I_4 - I_9$ ,  $I_{14} - I_{17}$ . Obviously,  $z_i$  will also contain moisture gradients.

We have:

$$z_i = F_{k\ell} \frac{\partial a_{k\ell}}{\partial x_i} + F_{k\ell}^* \frac{\partial a_{k\ell}^*}{\partial x_i} + F_m \frac{\partial m}{\partial x_i} \tag{45}$$

where 
$$F_{k\ell} = \frac{\partial}{\partial m} \left( \frac{1}{b_G} \frac{\partial A_0}{\partial a_{k\ell}} \right) + \frac{D'(\xi) t}{b_G} \frac{\partial}{\partial m} \left( \frac{b_G}{b} \right) \frac{\partial A_0}{\partial a_{k\ell}}$$

$$F_{kl}^* = F_{kl}(a_{kl}^*) \quad \text{and} \quad F_m = \frac{\partial^2}{\partial m^2} \left( \frac{A_o D(\xi)}{b_G} \right).$$

As can be noted from eqn. (45)  $z_i$  also depends on time through  $D(\xi)$  and its derivatives.

Combining equations (43) and (45) it is observed that the flux of moisture is reoriented into directions of increasing damage. Since damage growth can be attributed to moisture [Ashbee and Wyatt (1969), Drzal et. al (1985), Jackson and Weitsman (1985), and Weitsman (1987d)], the latter two observations would indicate that moisture absorption and damage form a synergistic mechanism.

#### 6. Concluding Remarks.

This paper presented a continuum-damage model for viscoelastic materials under several simplifying assumptions. The distinct micro-cracks were represented by two continuum-level internal state variables. These variables, which were expressed by non-dimensional symmetric, second-rank tensors, represented the total areas of "active" and "passive" micro-cracks contained within a representative volume element of the material. It was further assumed that all viscoelastic material responses follow a common time-dependence, and all detailed expressions were derived for materials that were isotropic prior to the onset of damage.

The resulting expressions show that, in the presence of damage, the response of viscoelastic materials contains two time-dependent phenomena. The first is attributable to the "inherent" visco-elastic behavior of the undamaged material, while the second is due to time-dependent damage growth. In addition, damage induces changes in the global symmetry of the material.

As can be noted from eqns. (37) - (39), the coupling of damage and viscoelasticity introduces very significant complexities in the material's

response and substantial work remains to be done - both experimentally and analytically - to attain a quantitative and practical understanding of the phenomenon.

The most important ingredient which remains missing in the present work concerns the growth relations for the damage parameters  $\dot{a}_{ij}$  and  $\dot{a}_{ij}^*$ . To be meaningful, such relations should be correlated with basic solutions for crack growth and closure in viscoelastic media. Several solutions for viscoelastic crack growth are already available at the present time, as reviewed by Kaminskii (1980) and outlined by Christensen (1982). Additional research into the fundamentals of viscoelastic fracture is necessary.

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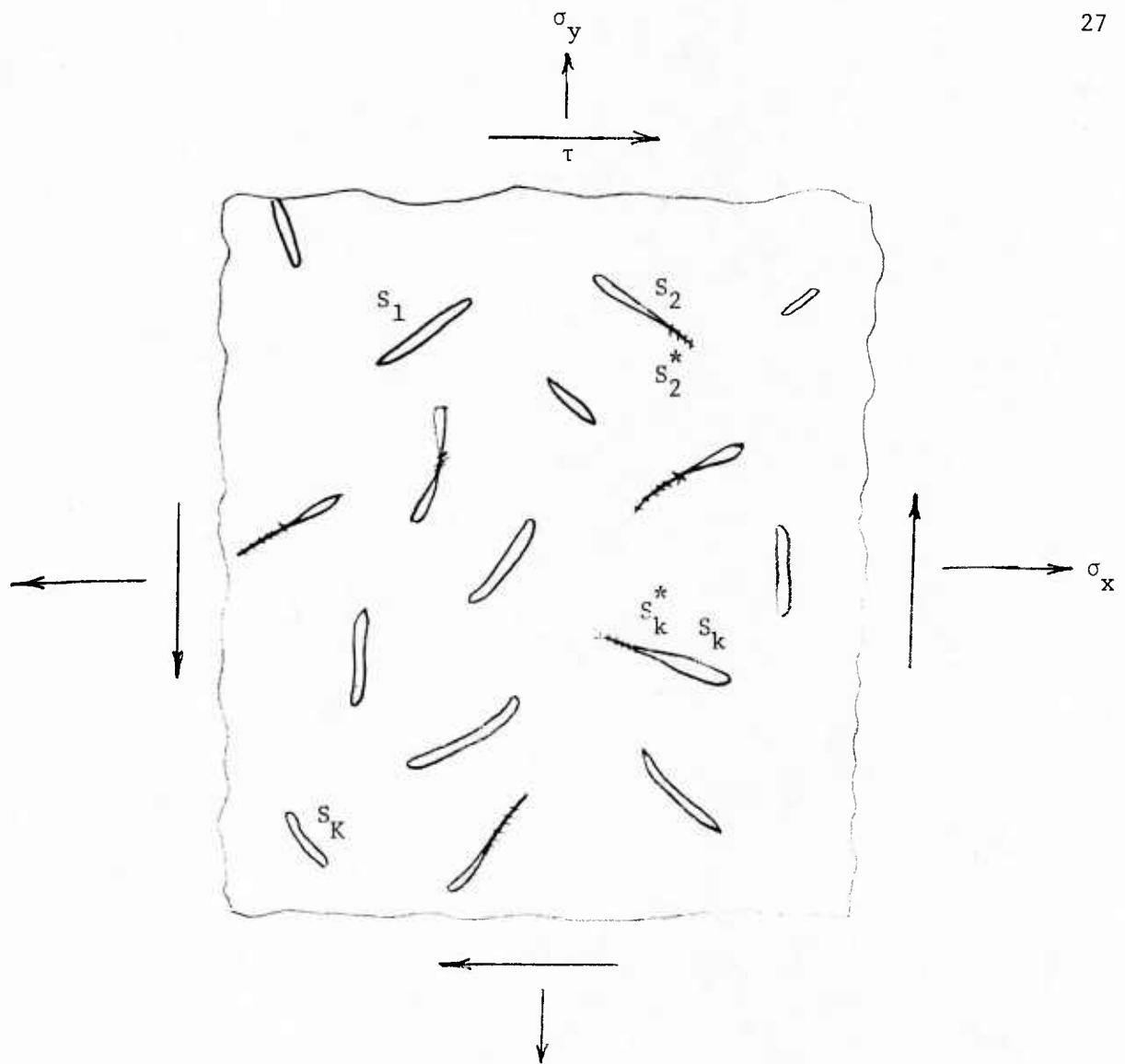


Fig. 1: A sketch of a Representative Volume Element Containing  $K$  Microcracks, with Open Surfaces  $S_k^*$  and Closed Portions  $S_k$ .